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THE ROLE OF BOUNDARY HARNACK PRINCIPLE
IN THE STUDY OF PICARD PRINCIPLE

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A nonnegative locally Hölder continuous function P on $0<|z|\leqq 1$ will be referred to as a *density* on $\Omega: 0<|z|<1$. A density on Ω gives rise to an elliptic operator L_P on Ω defined by

(1) $$L_P u = \Delta u - Pu, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

We say that the *Picard principle* (abbreviated as PP) is valid for P, rather for L_P, at $z=0$ if the dimension of the half module of nonnegative solutions of $L_P u=0$ on Ω with vanishing boundary values on $\partial\Omega - \{z=0\}$ is 1. With the operator L_P we associate an elliptic operator $\hat{L}_P$ on Ω, referred to as the *associate operator* to L_P, given by

(2) $$\hat{L}_P v = \Delta v + 2\nabla \log e_P \cdot \nabla v, \quad \nabla = (\partial/\partial x, \partial/\partial y),$$

where e_P, referred to as the P-*unit* on Ω, is the unique bounded solution of $L_P u=0$ on Ω with boundary values 1 on $\partial\Omega - \{z=0\}$. We also say that the *Riemann theorem* (abbreviated as RT) is valid for $\hat{L}_P$ at $z=0$ if the limit $\lim_{z\to 0} v(z)$ exists for every bounded solution v of $\hat{L}_P v=0$ on Ω. Then we have the *duality theorem* (cf. Heins [3], Hayashi [2], Nakai [8]): The Picard principle is valid for L_P at $z=0$ if and only if the Riemann theorem is valid for $\hat{L}_P$ at $z=0$. As a sufficient condition for the Riemann theorem for $\hat{L}_P$ at $z=0$ we have, what we call, the *boundary Harnack principle* (abbreviated as BHP) for L_P at $z=0$ (Kawamura [6]):

(3) $$\left\{\begin{array}{l}\text{For every Jordan region } U \text{ in } |z|<1 \text{ containing } z=0 \text{ there exists a} \\ \text{Jordan region } V_U \text{ containing } z=0 \text{ such that } \overline{V}_U \subset U \text{ and } u(\zeta)\leqq Cu(\xi) \text{ for} \\ \text{every nonnegative bounded solution } u \text{ of } L_P u=0 \text{ on } U-\{z=0\} \text{ and } \zeta,\xi \\ \text{in } \partial V_U, \text{ where } C \text{ is a positive constant independent of } U,u,\zeta, \text{ and } \xi.\end{array}\right.$$

In fact (3) implies the *boundary Harnack principle* for $\hat{L}_P$ at $z=0$ which is formulated in the same fashion as it is done for L_P originally considered by Kawamura [6] and then the Riemann theorem for $\hat{L}_P$ at $z=0$ is deduced from the boundary Harnack principle for $\hat{L}_P$ at $z=0$ ([6]). In short it has been known that the following string of implications holds:

(4) BHP for L_P $\Longrightarrow$ BHP for $\hat{L}_P$ $\Longrightarrow$ RT for $\hat{L}_P$ $\Longleftrightarrow$ PP for L_P.

The purpose of this lecture is to show that the Picard principle for L_P conversely implies the boundary Harnack principle for L_P ([9]). Therefore we can conclude that properties appearing in (4) are in fact all equivalent to each other :

(5) BHP for L_P $\Longleftrightarrow$ BHP for $\hat{L}_P$ $\Longleftrightarrow$ RT for $\hat{L}_P$ $\Longleftrightarrow$ PP for L_P.

We will also give an example of a density satisfying the boundary Harnack principle at $z=0$ ([9]) : If a density P on Ω satisfies $Q(z) \leqq P(z) \leqq Q(z) + C/|z|^2$ for a positive constant C and a *rotation free* density Q on Ω, i.e. a density satisfying $Q(z) = Q(|z|)$, for which the Picard principle is valid at $z=0$, then the boundary Harnack principle is valid for L_P at $z=0$ so that the Picard principle is valid for L_P at $z=0$.

1. The Harnack principle.

We will define a Harnack constant $C(K,\Omega_a,P)$ and deduce the *ordinary* Harnack principle. For a density P on Ω and a real number a in $(0,1]$ we denote by $G_P^{\Omega_a}$ the P-Green's function on $\Omega_a = \{0<|z|<a\}$, i.e. the Green's function on Ω_a with respect to the equation $L_P u = 0$. We consider a Harnack constant $C(K,\Omega_a,P)$ of a compact subset K of Ω_a defined by

$$C(K,\Omega_a,P) = \max \left\{ \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\xi)} \; ; \; |z| = a \text{ and } \zeta,\xi \text{ are in } K \right\},$$

where $\partial/\partial n_z$ means the inner normal derivative. Then the integral representation of a bounded solution of $L_P u = 0$ in terms of the inner normal derivative of the P-Green's function yields the following Harnack principle : for any nonnegative bounded solution u of $L_P u = 0$ on $\overline{\Omega}_a - \{z=0\}$ and ζ,ξ in K we have

$$u(\zeta) \leqq C(K,\Omega_a,P)u(\xi).$$

2. The boundary Harnack principle.

We will show that the Picard principle for L_P implies the boundary Harnack principle for L_P, and hence they are equivalent. Let P be a density on Ω such that the Picard principle is valid for L_P at $z=0$. Then the function $G_P^{\Omega_a}(z,\zeta)/e_P(\zeta)$ in z converges uniformly on every compact subset of $\overline{\Omega}_a - \{z=0\}$ as $\zeta \to 0$, and hence the inner normal derivative $\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)/e_P(\zeta)$ converges to a positive continuous function on $\partial\Omega_a - \{z=0\}$ (cf Itô [5]). In order to show (3) we consider two cases separately : $\limsup_{\zeta\to 0} e_P(\zeta) = 0$ and > 0.

First we consider the case $\limsup_{\zeta\to 0} e_P(\zeta) = 0$, i.e. $\lim_{\zeta\to 0} e_P(\zeta) = 0$. For every λ in $(0,1)$ let A_λ be a connected component of $\{\zeta \in \Omega \; ; \; e_P(\zeta) < \lambda\}$ such that $z = 0$ is an isolated boundary point of A_λ. Observe that $\overline{A}_\lambda \downarrow \{z=0\}$ as $\lambda \to 0$ and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)}{e_P(\zeta)} \; \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\xi)}$$

for ζ,ξ in $\partial A_\lambda - \{z=0\}$. Then we have

$$\lim_{\lambda\to 0} C(\partial A_\lambda - \{z=0\},\Omega_a,P) = 1$$

so that for every subregion U of $\{|z|<1\}$ containing $z=0$ we can take a_U, λ_U in $(0,1)$ with $\Omega_{a_U} \subset U$ and $C(\partial A_{\lambda_U} - \{z=0\},\Omega_{a_U},P) < 2$. Therefore (3) is valid for $C = 2$ and $V_U = A_{\lambda_U} \cup \{z=0\}$.

Assume next that $\limsup_{\zeta\to 0} e_P(\zeta) \equiv \delta > 0$. There exists a closed set E thin at $z = 0$ in Ω such that $e_P(\zeta) \to \delta$ as $\zeta \to 0$ with $\zeta \notin E$ (cf Brelot [1]). Then we can take a decreasing sequence $\{\lambda_n\}_1^\infty$ in $(0,1)$ with $E \cap \cup_1^\infty (\partial\Omega_{\lambda_n} - \{z=0\}) = \emptyset$ and $\lim \lambda_n = 0$. Observe that $e_P(\zeta) \to \delta$ as $\zeta \to 0$ with $\zeta \in \cup_1^\infty (\partial\Omega_{\lambda_n} - \{z=0\})$ and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\zeta)}{e_P(\zeta)} \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z,\xi)} \frac{e_P(\zeta)}{e_P(\xi)}$$

for ζ,ξ in $\partial\Omega_{\lambda_n} - \{z=0\}$. Then we have

$$\lim_{n\to\infty} C(\partial\Omega_{\lambda_n} - \{z=0\},\Omega_a,P) = 1.$$

Therefore (3) is valid for $C = 2$ and $V_U = \Omega_{\lambda_n} \cup \{z=0\}$ for some n depending on U.

3. Fundamental properties of units.

We now recall some of fundamental properties of the Q_n-unit. Let Q be a *rotation free* density on Ω, i.e. a density satisfying $Q(z) = Q(|z|)$. We consider a rotation free density $Q_n(z) = Q(z) + n^2/|z|^2$ on Ω for every nonnegative integer n and the Q_n-unit $f_n(z,a)$ on Ω_a, i.e. an unique bounded solution of $L_{Q_n} u = 0$ on Ω_a with boundary values 1 on $\partial\Omega_a - \{z=0\}$, where we follow the convention $Q_0 = Q$ and $f_0(z,1) = e_Q(z)$. Then $f_n(z,a)$ is rotation free and $f_n(r,a)$ is an unique bounded solution of

$$(6)\qquad \ell_n\psi(r) \equiv \ell_{Q_n}\psi(r) \equiv \frac{d^2}{dr^2}\psi(r) + \frac{1}{r}\frac{d}{dr}\psi(r) - Q_n(r)\psi(r) = 0$$

on $(0,a)$ with boundary values 1 at $r = a$. We have the following properties of $f_n(r,a)$ (cf Nakai [7]) :

$$(7)\qquad f_n(r,\rho) = \frac{f_n(r,a)}{f_n(\rho,a)} \quad (0 < r \leqq a,\ r \leqq \rho \leqq a);$$

$$(8)\qquad f_n(r,a) > f_{n+1}(r,a) \quad (0 < r < a);$$

$$(9)\qquad \frac{f_{n+1}(r,a)}{f_n(r,a)} \geqq \frac{f_{n+2}(r,a)}{f_{n+1}(r,a)} \quad (0 < r \leqq a);$$

$$(10)\qquad \left\{\frac{f_{n+1}(r,a)}{f_n(r,a)}\right\}^3 \leqq \frac{f_{n+2}(r,a)}{f_{n+1}(r,a)} \quad (0 < r \leqq a);$$

the Picard principle is valid for L_Q at $z = 0$ if and only if

$$(11)\qquad \lim_{r\to 0} \frac{f_1(r,a)}{f_0(r,a)} = 0$$

for some a, and hence by (7) any a in $(0,1]$. For another rotation free density R on Ω with $Q \leqq R$ we have also (cf Imai [4])

$$(12)\qquad \frac{f_{n+1}(r,a)}{f_n(r,a)} \leqq \frac{g_{n+1}(r,a)}{g_n(r,a)} \quad (0 < r \leqq a),$$

where $R_n(z) = R(z) + n^2/|z|^2$ and $g_n(z,a)$ is the R_n-unit on Ω_a $(n = 0, 1, \cdots)$.

4. Fourier coefficients of solutions.

We consider Fourier coefficients

$$\begin{cases} c_0(r,w) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} w(re^{i\theta})d\theta, \\[2ex] a_n(r,w) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} w(re^{i\theta})\cos n\theta\, d\theta, \end{cases}$$

$$\Big\lfloor\ b_n(r,w) = \frac{1}{\pi}\int_0^{2\pi} w(re^{i\theta})\sin n\theta\, d\theta$$

for a continuous function $w(z)$ on $\overline{\Omega}_a - \{z=0\}$. Here and hereafter let Q be a rotation free density on Ω and $f_n(z,a)$ the Q_n-unit on Ω_a. If w is further a bounded solution of $L_Q u = 0$ on Ω_a, then the Fourier coefficients of w are bounded solutions of (6) :

$$\ell_0 c_0(r,w) = \ell_n a_n(r,w) = \ell_n b_n(r,w) = 0.$$

Therefore they are represented in terms of Q_n-units :

$$\begin{cases} c_0(r,w) = c_0(a,w)f_0(r,a), \\ a_n(r,w) = a_n(a,w)f_n(r,a), \\ b_n(r,w) = b_n(a,w)f_n(r,a) \end{cases}$$

$(0 < r \leqq a\,;\, n = 1,2,\cdots)$.

5. Normal derivatives of Green's functions.

We expand the inner normal derivative of the Q-Green's function into its Fourier series. For any τ in $[0,2\pi)$ we denote by w_τ a bounded solution of $L_Q u = 0$ on Ω_a with boundary values 1 on $\{ae^{i\theta}\,;\, 0<\theta<\tau\}$ and 0 on $\{ae^{i\theta}\,;\, \tau<\theta<2\pi\}$. Then w_τ is represented in an integral form :

$$w_\tau(se^{i\sigma}) = \frac{1}{2\pi}\int_0^{\tau} \Big[-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\theta}, se^{i\sigma})\Big]_{r=a}\, a d\theta$$

for any $se^{i\sigma}$ in Ω_a. On the other hand w_τ is represented in a Fourier series :

$$w_\tau(se^{i\sigma}) = c_0(a,w_\tau)f_0(s,a)$$
$$+ \sum_{n=1}^{\infty} \{a_n(a,w_\tau)\cos n\sigma + b_n(a,w_\tau)\sin n\sigma\} f_n(s,a).$$

Since by (8) and (9) we have

$$\text{(13)}\qquad \frac{f_1(s,a)}{f_0(s,a)} < 1, \quad f_n(s,a) \leqq f_0(s,a)\left\{\frac{f_1(s,a)}{f_0(s,a)}\right\}^n,$$

we obtain

$$\frac{\partial}{\partial\tau} w_\tau(se^{i\sigma}) = \frac{\partial}{\partial\tau} c_0(a,w_\tau)f_0(s,a)$$

$$+ \sum_{n=1}^{\infty} \frac{\partial}{\partial\tau} \{a_n(a,w_\tau)\cos n\sigma + b_n(a,w_\tau)\sin n\sigma\}f_n(s,a).$$

Observe that

$$\frac{\partial}{\partial\tau} c_0(a,w_\tau) = \frac{\partial}{\partial\tau}\frac{1}{2\pi}\int_0^\tau d\theta = \frac{1}{2\pi},$$

$$\frac{\partial}{\partial\tau} a_n(a,w_\tau) = \frac{\partial}{\partial\tau}\frac{1}{\pi}\int_0^\tau \cos n\theta\, d\theta = \frac{1}{\pi}\cos n\tau,$$

and

$$\frac{\partial}{\partial\tau} b_n(a,w_\tau) = \frac{1}{\pi}\sin n\tau.$$

Then we expand the inner normal derivative of the Q-Green's function into the following Fourier series :

$$[-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a} = \frac{1}{a}\{f_0(s,a) + 2\sum_{n=1}^{\infty} f_n(s,a)\cos n(\sigma-\tau)\}.$$

Estimating the right hand side of this equality by using (13) we have the following inequalities :

$$\text{(14)}\qquad [-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a} \leqq \frac{1}{a} f_0(s,a)\left\{1+\frac{f_1(s,a)}{f_0(s,a)}\right\}\left\{1-\frac{f_1(s,a)}{f_0(s,a)}\right\}^{-1}$$

and

$$\text{(15)}\qquad [-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a} \geqq \frac{1}{a} f_0(s,a)\left\{1-3\frac{f_1(s,a)}{f_0(s,a)}\right\}\left\{1-\frac{f_1(s,a)}{f_0(s,a)}\right\}^{-1}.$$

6. The Picard principle.

We give an example of a density on Ω satisfying the boundary Harnack principle, and hence the Picard principle. Let P be a general and Q a rotation free density on Ω such that the Picard principle is valid for L_Q at $z=0$ and

$$Q(z) \leqq P(z) \leqq Q(z) + \frac{C}{|z|^2}$$

for a positive constant C. We take a positive integer k with $9k^2 > C$ and consider a rotation free density $R(z) = Q(z) + 9k^2/|z|^2$ on Ω. First we evaluate the inner normal derivative of the P-Green's function in terms of Q_n-unit $f_n(z,a)$ and R_n-unit $g_n(z,a)$ on Ω_a. Since the P-Green's function satisfies

$$G_R^{\Omega_a} \leqq G_P^{\Omega_a} \leqq G_Q^{\Omega_a}$$

we have

$$\begin{aligned}[-\frac{\partial}{\partial r} G_R^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a} &\leqq [-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a} \\ &\leqq [-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau},se^{i\sigma})]_{r=a}\end{aligned}$$

for every τ in $[0,2\pi)$ and $se^{i\sigma}$ in Ω_a. Then by (12), (14), and (15) we obtain

$$(16) \qquad \frac{[-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau},se^{i\alpha})]_{r=a}}{[-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau},se^{i\beta})]_{r=a}} \leqq \frac{f_0(s,a)}{g_0(s,a)} \frac{1+\dfrac{g_1(s,a)}{g_0(s,a)}}{1-3\,\dfrac{g_1(s,a)}{g_0(s,a)}}$$

for any α,β in $[0,2\pi)$ if $g_1(s,a)/g_0(s,a) < 1/3$.

Next we evaluate $f_0(s,a)/g_0(s,a)$ in terms of $g_1(s,a)/g_0(s,a)$. From (10) it follows that

$$(17) \qquad \frac{g_{4k}(s,a)}{g_0(s,a)} \geqq \left\{ \frac{g_1(s,a)}{g_0(s,a)} \right\}^{(81^k-1)/2}$$

and

$$\frac{f_{3k}(s,a)}{f_0(s,a)} \geqq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{(27^k-1)/2}. \tag{18}$$

Observe that $g_0 = f_{3k}$ and $g_{4k} = f_{5k}$. Then (17), (18), and

$$\frac{f_{5k}(s,a)}{f_{3k}(s,a)} \leqq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{2k} \tag{19}$$

yield an evaluation

$$\frac{f_0(s,a)}{g_0(s,a)} \leqq \left\{ \frac{g_0(s,a)}{g_1(s,a)} \right\}^{\alpha_k},$$

where $\alpha_k = (81^k - 1)(27^k - 1)/8k$.

Now we show the boundary Harnack principle (3) for L_P at $z = 0$. We have by (17) and (19)

$$\frac{g_1(s,a)}{g_0(s,a)} \leqq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{4k/(81^k-1)}.$$

Then by (11) we can take s_a in $(0,a)$ such that $g_1(s_a,a)/g_0(s_a,a) = 1/4$. Therefore by (16) we obtain

$$C(\partial\Omega_{s_a} - \{z=0\}, \Omega_a, P) \leqq 5 \cdot 4^{\alpha_k}.$$

Thus (3) is valid for $C = 5 \cdot 4^{\alpha_k}$ and $V_U = \Omega_{s_a} \cup \{z=0\}$, where a is a positive number with $\Omega_a \subset U$, so that the Picard principle is valid for L_P at $z = 0$.

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